Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(02)00088-6

J. Appl. Maths Mechs, Vol. 66, No. 4, pp. 673–682, 2002

0021-8928/02/\$—see front matter

EXACT SOLUTIONS OF SOME MIXED PROBLEMS OF UNCOUPLED THERMOELASTICITY THEORY FOR A FINITE HOLLOW CIRCULAR CYLINDER WITH A GROOVE ALONG THE GENERATRIX†

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(*Received* 27 *June* 2001)

Uncoupled problems of thermoelasticity for a finite hollow circular cylinder with a groove along the generatrix for different types of boundary conditions on all the surfaces are considered. These are the conditions for specifying displacements equal to zero or sliding clamping on the cylindrical surfaces and on the edges of the groove and the conditions for specifying stresses on the ends of the cylinder (shear stresses are assumed to be zero). It is assumed that the problem of thermoelasticity has been solved and the temperature is known. Certain auxiliary functions, related to the displacements, are initially introduced, and equations for these functions are derived using Lamé's equations. A finite integral Fourier transformation with respect to the polar angle and a finite Hankel transformation with respect to the radius are then used. As a result a one-dimensional boundary-value vector problem is obtained for a first-order differential equation, which is solved using Green's matrix constructed. Finally, exact solutions of the problems are constructed in series. © 2002 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEMS

We will consider steady problems of thermoelasticity [1, 2] for part of a finite hollow circular cylinder, which, in a cylindrical system of coordinates (r, φ, z), occupies the region

$$a_0 \leq r \leq a_1, \quad \varphi_0 \leq \varphi \leq \varphi_1, \quad h_0 \leq z \leq h_1 \tag{1.1}$$

On the cylindrical surfaces $r = a_i$ $(i = 0, 1)$ and on the faces $\varphi = \varphi_i$ $(i = 0, 1)$ the displacements u_r, u_φ and u_z or the conditions for sliding closure are specified, i.e.

$$\begin{gathered} u_r(a_i, \varphi, z) = 0, \quad \tau_{r\varphi}(a_i, \varphi, z) = \tau_{rz}(a_i, \varphi, z) = 0; \quad i = 0, 1 \\ u_\varphi(r, \varphi_i, z) = 0, \quad \tau_{\varphi r}(r, \varphi_i, z) = \tau_{z\varphi}(r, \varphi_i, z) = 0; \quad i = 0, 1 \end{gathered} \tag{1.2}$$

On the end faces $z = h_i$ $(i = 0, 1)$ the boundary conditions may be arbitrary, but to fix our ideas we will take them to be as follows:

$$\sigma_z(r, \varphi, h_1) = -p^{(i)}(r, \varphi), \quad \tau_{rz}(r_i, \varphi, h_i) = \tau_{z\varphi}(r_i, \varphi, h_i) = 0; \quad i = 0, 1 \tag{1.3}$$

In the case of steady uncoupled thermoelasticity, the temperature field can be found in advance by solving one or other boundary-value problem for Laplace's equation. Since similar problems have been fairly well investigated for cylindrical bodies [3], we will assume that the temperature field is already known. Another approach to solving boundary-value problems of thermoelasticity for hollow circular cylinders was proposed in [4] for the case when there are no faces $\varphi = \varphi_i$ $(i = 0, 1)$.

Note that the problem formulated in this way is equivalent to the problem of solving the similar boundary-value problem for inhomogeneous Lamé equations with volume forces of a special form.

2. CONVERSION OF THE THERMOELASTICITY EQUATIONS TO A NEW FORM

We will introduce the following notation for the displacements in a cylindrical system of coordinates

$$2G\|u_r, u_\varphi, u_z\| = \|U, V, W\| \tag{2.1}$$

†*Prikl. Mat. Mekh.* Vol. 66, No. 4, pp. 694–704, 2002.

where G is the shear modulus, and also the functions $Z(r, \varphi, z)$ and $Z^*(r, \varphi, z)$, defined by the formulae

$$\begin{Vmatrix} Z \\ Z^* \end{Vmatrix} = \frac{1}{r}\left\{\begin{Vmatrix} rU \\ rV \end{Vmatrix}' \pm \begin{Vmatrix} V^{\cdot} \\ U^{\cdot} \end{Vmatrix}\right\} \tag{2.2}$$

Then the thermoelasticity equations can be written in the form [1, 2]

$$\Delta U - r^{-2}(U + 2V^{\cdot}) + \mu_0 \tilde{Z}' = \alpha_\mu T'$$

$$\Delta V - r^{-2}(V - 2U^{\cdot}) + r^{-1}\mu_0 \tilde{Z}^{\cdot} = \alpha_\mu r^{-1} T^{\cdot}$$

$$\Delta W + \mu_0 \tilde{Z}_{,} = \alpha_\mu T_{,}$$

$$\mu_0 = (1 - 2\mu)^{-1}, \quad \alpha_\mu = 4\alpha_T G(1 + \mu)\mu_0, \quad \tilde{Z} = Z + W_{,} \tag{2.3}$$

Here μ is Poisson's ratio, α_T is the coefficient of linear expansion, $T = T(r, \varphi, z)$ is the temperature and Δ is the Laplace operator in a cylindrical system of coordinates; the partial derivatives with respect to r are denoted by a prime, partial derivatives with respect to the variable φ are denoted by a point and partial derivatives with respect to the variable z are denoted by a subscript comma.

We will convert Eqs (2.3), for which we multiply the first equation by r, we differentiate with respect to r and divide by r, and we differentiate the second equations with respect to φ and divide by r. We add the equations obtained and then carry out the same operation with the second equation of (2.3) as on the first equation of (2.3) in the previous case, while on the first equation of (2.3) we carry out the same operation as on the second equation in the previous case, and subtract the results. We finally reduce Eqs (2.3) to the form

$$\Delta W + \mu_0 \tilde{Z}_{,} = \alpha_\mu T_{,}, \quad \Delta Z + \mu_0 \tilde{Z} = \alpha_\mu \nabla T, \quad \Delta Z^* = 0 \tag{2.4}$$

$$\nabla f(r, \varphi, z) = r^{-1}[rf'[r, \varphi, z)]' + r^{-2} f^{\cdot\cdot}(r, \varphi, z)$$

If we solve Eqs (2.4), i.e. obtain the functions Z and Z^*, defined by formulae (2.2), then to find the functions U and V, defining the displacements, we must solve the equations

$$\nabla \begin{Vmatrix} rU \\ rV \end{Vmatrix} = \frac{1}{r}\begin{Vmatrix} (r^2 Z)' \\ (r^2 Z^*)' \end{Vmatrix} \mp \begin{Vmatrix} Z^{*\cdot} \\ Z^{\cdot} \end{Vmatrix} \tag{2.5}$$

They are obtained from (2.2) using operations similar to those carried out on Eqs (2.3) when obtaining Eqs (2.4).

3. INTEGRAL TRANSFORMATION OF THE EQUATIONS OBTAINED WITH RESPECT TO THE VARIABLE φ

The integral transformation of the equations obtained depends very much on what conditions are imposed on the faces $\varphi = \varphi_i$ $(i = 0, 1)$. If the displacements are specified on them, we use a Fourier sine transformation in a finite interval, assuming in this case that $\varphi_0 = 0$, i.e. we change in Eqs (2.4) to the transforms

$$X_n(r, z) = \int_0^{\varphi_1} X(r, \varphi, z)\sin\mu_n\varphi d\varphi, \quad \mu_n = \frac{n\pi}{\varphi_1}, \quad n = 1, 2, \ldots \tag{3.1}$$

(the transforms Z_n^* and T_n are defined similarly). Here we have introduced the following notation

$$X_n(r, z) = \begin{Vmatrix} W_n(r, z) \\ Z_n(r, z) \end{Vmatrix}, \quad X(r, \varphi, z) = \begin{Vmatrix} W(r, \varphi, z) \\ Z(r, \varphi, z) \end{Vmatrix} \text{ etc.}$$

The inversion formulae for the transforms have the form [5]

$$X(r, \varphi, z) = \frac{2}{\varphi_1}\sum_{n=1}^{\infty} X_n(r, z)\sin\mu_n\varphi, \quad \mu_n = \frac{n\pi}{\varphi_1}, \quad n = 1, 2, \ldots \tag{3.2}$$

If the conditions of sliding clamping are specified on the faces $\varphi = \varphi_i$ ($i = 0, 1$), i.e. conditions (1.2), then instead of integral transformation (3.1) we must take the following

$$X_n(r,z) = \int_0^{\varphi_1} X(r,\varphi,z)\cos\mu_n\varphi d\varphi, \quad \mu_n = \frac{(n-1)\pi}{\varphi_1}, \quad n = 1,2,\ldots \tag{3.3}$$

with the inversion formulae [5]

$$X(r,\varphi,z) = \frac{2}{\varphi_1}\sum_{n=1}^{\infty}{}' X_n(r,z)\cos\mu_n\varphi, \quad \mu_n = \frac{(n-1)\pi}{\varphi_1}, \quad n = 1,2,\ldots \tag{3.4}$$

(the prime on the summation sign denotes that the first term must be multiplied by $1/2$).

The use of integral transformation (3.1) denotes that the following boundary conditions are satisfied

$$W(r,\varphi_i,z) = Z(r,\varphi_i,z) = Z^*(r,\varphi_i,z) = 0, \quad i = 0,1 \tag{3.5}$$

and the use of second integral transformation denotes that the following boundary conditions are satisfied

$$W'(r,\varphi_i,z) = Z'(r,\varphi_i,z) = Z^{*'}(r,\varphi_i,z) = 0, \quad i = 0,1 \tag{3.6}$$

If $\varphi_0 = -\pi$ and $\varphi_1 = \pi$ in conditions (1.1), i.e. there are no faces $\varphi = \varphi_i$ ($i = 0, 1$), then, instead of integral transformations (3.1) and (3.3), we must take the following integral transformation [3]

$$X_n(r,z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(r,\varphi,z)e^{-in\varphi}d\varphi, \quad n = 0,\pm1,\pm2,\ldots \tag{3.7}$$

with the inversion formulae [3]

$$X(r,\varphi,z) = \sum_{n=-\infty}^{\infty} X_n(r,z)e^{in\varphi} \tag{3.8}$$

If we apply integral transformations (3.1), (3.3) and (3.7) to Eqs (2.4) they acquire the form

$$W_n^{\cdot\cdot} - \mu_*^{-1}\nabla_n W_n + \mu_0\mu_*^{-1}Z_n^{\cdot} = \alpha_\mu\mu_*^{-1}T_n^{\cdot}, \quad Z_n^{\cdot\cdot} - \mu_*\nabla_n Z_n + \mu_0\nabla_n W^{\cdot} = -\alpha_\mu\nabla_n T_n; \tag{3.9}$$

$$\mu_* = 2(1-\mu)\mu_0 \quad Z_n^{*\cdot\cdot} - \nabla_n Z_n^* = 0$$

where

$$\nabla_n f(r,z) = \mu_n^2 r^{-2} f(r,z) - r^{-1}[rf'(r,z)]' \tag{3.10}$$

Here, in the case of integral transformation (3.1) the expression for μ_n is taken from (3.1), in the case of integral transformation (3.3) the expression for μ_n is taken from (3.3), while in the case of integral transformation (3.7) $\mu_n = n$ ($n = 0, \pm1, \pm2, \ldots$).

We make the following replacement in Eqs (3.9)

$$X_n(r,z) = X_n(a_1\rho, a_1\zeta) = \overline{X}_n(\rho,\zeta) \tag{3.11}$$

$$Z_n^*(r,z) = Z_n^*(a_1\rho, a_1\zeta) = \overline{Z}_n^*(\rho,\zeta), \quad T_n(r,z) = T_n(a_1\rho, a_1\zeta) = \overline{T}_n(\rho,\zeta)$$

As a result we obtain

$$\overline{W}_n^{\cdot\cdot}(\rho,\zeta) - \mu_*^{-1}\nabla_n\overline{W}_n(\rho,\zeta) + a_1\mu_0\mu_*^{-1}\overline{Z}_n^{\cdot}(\rho,\zeta) = a_1\alpha_\mu\mu_*^{-1}\overline{T}_n^{\cdot}(\rho,\zeta)$$

$$\overline{Z}^{\cdot\cdot}(\rho,\zeta) - \mu_*\nabla_n\overline{Z}_n(\rho,\zeta) - a_1^{-1}\mu_0\nabla_n\overline{w}_n^{\cdot}(\rho,\zeta) = -\alpha_\mu\nabla_n\overline{T}_n(\rho,\zeta) \tag{3.12}$$

$$\overline{Z}_n^{*\cdot\cdot}(\rho,\zeta) - \nabla_n\overline{Z}_n^*(\rho,\zeta) = 0$$

$$a = a_0/a_1, \quad \overline{h}_1 = a_1^{-1}h_i, \quad i = 0,1, \quad \overline{h}_0 < \zeta < \overline{h}_1$$

Here and everywhere henceforth a dot denotes a partial derivative with respect to the variable ζ, and a prime denotes a partial derivative with respect to ρ.

4. REDUCTION OF THE EQUATIONS TO ONE-DIMENSIONAL EQUATIONS

To reduce Eqs (3.12) to one-dimensional equations we will use an integral transformation with respect to the variable ρ while satisfying the conditions

$$\overline{W}_n(\rho_i, \zeta) = \overline{Z}_n(\rho_i, \zeta) = \overline{Z}_n^{*\prime}(\rho_i, \zeta) = 0, \quad i = 0, 1 \ (\rho_0 = a, \ \rho_1 = 1) \tag{4.1}$$

for the case when the displacements are specified on the faces $r = a_i$ and $\varphi = \varphi_i$ ($i = 0, 1$), while satisfying the conditions

$$\overline{W}_n'(\rho_i, \zeta) = \overline{Z}_n'(\rho_i, \zeta) = \overline{Z}_n^{*\prime}(\rho_i, \zeta) = 0, \quad i = 0, 1 \ (\rho_0 = a, \ \rho_1 = 1) \tag{4.2}$$

when conditions (1.2) are specified, i.e. in Eqs (3.12) we change to the transforms [3]

$$\overline{X}_{nk}(\zeta) = \int_a^1 \overline{X}_n(\rho, \zeta)\varphi_{n,j}(\rho, \nu)\rho d\rho, \quad j = 0, 1; \quad \nu = \nu_k^j, \quad k = 0, 1, \ldots \tag{4.3}$$

Here

$$\begin{aligned} &\varphi_{n,0} = N_\mu(a\nu)J_\mu(\nu\rho) - J_\mu(a\nu)N_\mu(\nu\rho); \quad \mu = \mu_n = n\pi\varphi_1^{-1}, \quad \nu = \nu_k^0 \\ &\varphi_{n,1} = N_\mu'(a\nu)J_\mu(\nu\rho) - J_\mu'(a\nu)N_\mu(\nu\rho); \quad \mu = \mu_n = (n-1)\pi\varphi_1^{-1}, \quad \nu = \nu_k^1; \quad n, \ k = 1, 2, \ldots \end{aligned} \tag{4.4}$$

($N_\mu(x)$ is the Neumann function and $J_\mu(x)$ is the Bessel function). The transforms $\overline{Z}_{nk}^*(\zeta)$ and $\overline{T}_{nk}(\zeta)$ are defined similarly.

The eigenvalues ν_k^0 and ν_k^1 are the positive roots of the transcendental equations

$$\begin{aligned} &N_\mu(a\nu)J_\mu(\nu) - J_\mu(a\nu)N_\mu(\nu) = 0; \quad \mu = \mu_n = n\pi\varphi_1^{-1}, \quad \nu = \nu_k^0 \\ &N_\mu'(a\nu)J_\mu'(\nu) - J_\mu'(a\nu)N_\mu'(\nu) = 0; \quad \mu = \mu_n = (n-1)\pi\varphi_1^{-1}, \quad \nu = \nu_k^1; \quad n, k = 1, 2, \ldots \end{aligned} \tag{4.5}$$

The inversion formulae for transforms (4.3) have the form [3]

$$\overline{X}_n(\rho, \zeta) = \sum_{k=1}^{\infty} \overline{X}_{nk}(\zeta)\frac{\varphi_{n,j}(\rho, \nu)}{\left\|\varphi_{n,j}(\rho, \nu)\right\|^2}, \quad j = 0, 1; \quad \nu = \nu_k^j, \quad k = 0, 1 \tag{4.6}$$

where

$$\begin{aligned} &2\left\|\varphi_{n,0}(\rho, \nu)\right\|^2 = [\varphi_{n,0}'(\nu, \nu)]^2 - a^2[\varphi_{n,0}'(\nu a, \nu)], \quad \nu = \nu_k^0, \quad \mu = \mu_n = n\pi\varphi_1^{-1} \\ &2\nu^2\left\|\varphi_{n,1}(\rho, \nu)\right\|^2 = (\nu^2 - \mu^2)\varphi_{n,1}^2(\nu, \nu) - (\nu^2 a^2 - \mu^2)\varphi_{n,1}(\nu a, \nu), \quad \nu = \nu_k^1, \\ &\mu = \mu_n = (n-1)\pi\varphi_1^{-1}; \quad n, k = 1, 2, \ldots \end{aligned} \tag{4.7}$$

The functions $\varphi_{n,j}(\rho, \nu)$ satisfy Bessel's equation, i.e.

$$\nabla_n\varphi_{n,j}(\rho, \nu) = \nu^2\varphi_{n,l}(\rho, \nu), \quad j = 0, 1; \quad \nu = \nu_k^j, \quad k = 1, 2, \ldots \tag{4.8}$$

By applying integral transformation (4.3) to Eqs (3.12) and taking Eq. (4.8) into account we reduce them to the form

$$\begin{aligned} &\overline{W}_{nk}''(\zeta) - \frac{\nu^2}{\mu_*}\overline{W}_{nk}(\zeta) + \frac{\mu_0 a_1}{\mu_*}\overline{Z}_{nk}'(\zeta) = \frac{\alpha_\mu a_1}{\mu_*}\overline{T}_{nk}'(\zeta) \\ &\overline{Z}_{nk}''(\zeta) - \mu_*\nu^2\overline{Z}_{nk}(\zeta) - \frac{\mu_0\nu^2}{a_1}\overline{W}_{nk}'(\zeta) = -\alpha_\mu\nu^2\overline{T}_{nk}(\zeta) \\ &\overline{Z}_{nk}^{*\prime\prime}(\zeta) - \nu^2\overline{Z}_{nk}^*(\zeta) = 0; \quad \overline{h}_0 < \zeta < \overline{h}_1 \end{aligned} \tag{4.9}$$

Here, when applying integral transformation (4.3) to Eqs (3.12) with $j = 0$ (conditions (4.1) are satisfied) $\nu = \nu_k^0(k = 1, 2, \ldots)$, and when $j = 1$ in relations (4.3) (conditions (4.2) are satisfied), $\nu = \nu_k^1(k = 1, 2, \ldots)$. If $\varphi_0 = -\pi$ and $\varphi_1 = \pi$ in conditions (1.1), then the parameter μ in formulae (4.4)–(4.10) must be fixed by the expression $\mu = \mu_n = n$.

5. CONSTRUCTION OF THE SOLUTION OF THE ONE-DIMENSIONAL DIFFERENTIAL EQUATIONS

The solution will be constructed so as to satisfy conditions (1.3). In order to write the latter in terms of the functions (2.2), we will consider combinations for the shear stresses τ_{zr} and $\tau_{z\varphi}$, similar to (2.2)

$$\left\|\begin{matrix}\tau(r,\varphi,z)\\ \tau^*(r,\varphi,z)\end{matrix}\right\| = \frac{1}{r}\left\{\left\|\begin{matrix}(r\tau_{rz})'\\ (r\tau_{z\varphi})'\end{matrix}\right\| \pm \left\|\begin{matrix}\tau_{z\varphi}^{\cdot}\\ \tau_{zr}^{\cdot}\end{matrix}\right\|\right\} \tag{5.1}$$

Using the relation between the stresses and displacements given by Hooke's law in a cylindrical system of coordinates [2], we obtain

$$2\tau = \nabla W + Z^{\cdot}, \quad 2\tau^* = Z^{*\cdot}$$

or, after applying the integral transformations given by formulae (3.1) and (3.7) and the replacement of variables given by (3.11),

$$2a_1^2\bar{\tau}_n(\rho,\zeta) = a_1\bar{Z}_n' - \nabla_n\bar{W}_n, \quad 2a_1\bar{\tau}_n^*(\rho,\zeta) = \bar{Z}_n^{*\cdot} \tag{5.2}$$

The last two conditions of (1.3), in view of relations (5.1), become $\tau(r, \varphi, h_i) = \tau^*(r, \varphi, h_i) = 0$ $(i = 0, 1)$, which, according to Eqs (5.2), are equivalent to the conditions

$$a_1\bar{Z}_n^{\cdot}(\rho,\bar{h}_i) - \nabla_n\bar{W}_n(\rho,\bar{h}_i) = 0, \quad \bar{Z}_n^{*\cdot}(\rho,\bar{h}_i) = 0; \quad i = 0, 1 \tag{5.3}$$

As a result, by virtue of relations (3.5), (3.6), (4.1), (4.2) and (5.3) we arrive at a homogeneous boundary-value problem for the function $\bar{Z}_n^*(\rho, \zeta)$, and therefore

$$\bar{Z}_n^*(\rho,\zeta) = 0 \tag{5.4}$$

Applying integral transformation (4.3) to the second equation of (5.3), we obtain

$$a_1^{-1}\nu^2\bar{W}_n(\bar{h}_i) - \bar{Z}_{nk}'(\bar{h}_i) = 0, \quad i = 0, 1 \tag{5.5}$$

Since the normal stress σ_z by virtue of Hooke's law [2], and taking Eqs (2.2) into account, is expressed by the formula

$$(1-2\mu)\sigma_z(r,\varphi,z) = \mu Z + (1-\mu)W^{\cdot} - (1-2\mu)\alpha_\mu T$$

the first boundary condition of (1.3), taking (3.11) into account as well as the integral transformations indicated above, can be written as

$$\begin{gathered} a_1^{-1}(1-\mu)\bar{W}_{nk}'(\bar{h}_i) + \mu\bar{Z}_{nk}(\bar{h}_i) = -(1-2\mu)g_{nk}^{(i)} \\ g_{nk}^{(i)} = p_{nk}^{(i)} - \alpha_\mu\bar{T}_{nk}(\bar{h}_i); \quad i = 0, 1 \end{gathered} \tag{5.6}$$

If we introduce the following functions as the unknowns of system (4.9)

$$y_0(\zeta) = \bar{W}_{nk}(\zeta), \quad y_1(\zeta) = \bar{W}_{nk}'(\zeta), \quad y_2(\zeta) = \bar{Z}_{nk}(\zeta), \quad y_3(\zeta) = \bar{Z}_{nk}'(\zeta) \tag{5.7}$$

which are the components of the required vector $y(\zeta)$, this system can be written in the following vector form

$$\mathbf{y}'(\zeta) - \mathbf{P}_k\mathbf{y}(\zeta) = \mu_*^{-1}\alpha_\mu\mathbf{f}(\zeta), \quad \bar{h}_0 < \zeta < \bar{h}_1 \tag{5.8}$$

where

$$\mathbf{y}(\zeta)=\begin{Vmatrix}\mathbf{y}_0(\zeta)\\ \mathbf{y}_1(\zeta)\\ \mathbf{y}_2(\zeta)\\ \mathbf{y}_3(\zeta)\end{Vmatrix},\quad \mathbf{P}_k=\begin{Vmatrix}0 & 1 & 0 & 0\\ \mu_k^{-1}\nu^2 & 0 & 0 & -\mu_*^{-1}a_1\mu_0\\ 0 & 0 & 0 & 1\\ 0 & a_1^{-1}\mu_0\nu^2 & \mu_*\nu^2 & 0\end{Vmatrix},\quad \mathbf{f}(\xi)=\begin{Vmatrix}0\\ a_1\overline{T}'_{nk}(\zeta)\\ 0\\ -\mu_*\nu^2\overline{T}_{nk}(\zeta)\end{Vmatrix} \tag{5.9}$$

In order to relate the boundary conditions to differential equation (5.8), we construct the boundary functional

$$\mathbf{U}[\mathbf{y}(\zeta)]=\mathbf{A}\mathbf{y}(\overline{h}_0)+\mathbf{B}\mathbf{y}(\overline{h}_1) \tag{5.10}$$

are the vector γ, defined by the formulae

$$\mathbf{A}=\begin{Vmatrix}a_1^{-1}\nu^2 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\\ 0 & a_1^{-1}(1-\mu) & \mu & 0\\ 0 & 0 & 0 & 0\end{Vmatrix},\quad \gamma=\begin{Vmatrix}0\\ 0\\ g_{nk}^{(0)}\\ g_{nk}^{(1)}\end{Vmatrix} \tag{5.11}$$

(the matrix **B** is obtained from the matrix **A** by interchanging the first two and the last two rows). Then, in order to satisfy conditions (5.5) and (5.6), written taking the notation (5.7) into account, we must add the following boundary conditions to Eq. (5.8)

$$\mathbf{U}[\mathbf{y}(\zeta)]=-(1-2\mu)\gamma \tag{5.12}$$

6. SOLUTION OF THE VECTOR BOUNDARY-VALUE PROBLEM

Extending the right-hand side of Eq. (5.8) to zero along the whole real axis, we apply a Fourier transformation to it

$$\mathbf{y}_\alpha=\int_{-\infty}^{\infty}\mathbf{y}(\zeta)e^{i\alpha\zeta}d\zeta$$

We obtain the solution in the form

$$\mathbf{y}(\zeta)=\int_{-\infty}^{\infty}\mathbf{f}(\zeta)\mathbf{\Phi}(\zeta-s)ds,\quad \mathbf{\Phi}(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[-i\alpha\mathbf{I}-\mathbf{P}_k]^{-1}e^{-i\alpha y}d\alpha$$

where $\mathbf{\Phi}(y)$ is the fundamental matrix function [6, 7] of inhomogeneous equation (5.8). It can also be written in the form

$$\mathbf{\Phi}(y)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}[\xi\mathbf{I}-\mathbf{P}_k]^{-1}e^{\xi y}d\xi,\quad y=\zeta-s \tag{6.1}$$

It follows from the expression for the characteristic polynomial of the matrix $\mathbf{P}_k$ [8, 7]

$$\det(\xi\mathbf{I}-\mathbf{P}_k)=Q_4(\xi)=\xi^4-2\xi^2\nu^2+\nu^4=(\xi^2-\nu^2)^2 \tag{6.2}$$

that is has two multiple roots: ν and $-\nu$. Then, the added matrix $\Delta^*(\xi)$ for the matrix $\mathbf{P}_k$ can be represented in the form [8, 7]

$$\Delta^*(\xi)=\sum_{j=0}^{3}\xi^j\Delta_{3-j} \tag{6.3}$$

$$\Delta_0=\mathbf{I},\ \Delta_1=\mathbf{P}_k,\ \Delta_2=\mathbf{P}_k^2-2\nu^2\mathbf{P}_k,\ \Delta_3=\mathbf{P}_k^3-2\nu^2\mathbf{P}_k=-\nu^4\mathbf{P}_k^{-1}$$

and we obtain the equality

$$(\xi\mathbf{I}-\mathbf{P}_k)^{-1}=Q_4^{-1}\sum_{j=0}^{3}\xi^j\Delta_{3-j} \tag{6.4}$$

Substituting (6.4) into (6.1), we obtain

$$\boldsymbol{\Phi}(y)=\sum_{j=0}^{3}\Delta_{3-j}\left(\frac{d}{dy}\right)^{j}\psi(y),\quad y=\zeta-s \tag{6.5}$$

$$\psi(y)=\frac{1}{2\pi i}\sum_{-i\infty}^{+i\infty}\frac{1}{(\xi-\nu)^{2}}e^{\xi y}\frac{d\xi}{(\xi+\nu)^{2}}=\frac{\nu|y|+1}{4\nu^{3}}e^{-\nu|y|} \tag{6.6}$$

(relation (6.2) has been taken into account).

Since the boundary conditions of the boundary-value problem are inhomogeneous, to solve it not only is Green's matrix function [6, 7] necessary, but also the basis matrix function $\boldsymbol{\Psi}(\zeta)$, which is the solution of the matrix boundary-value problem

$$\boldsymbol{\Psi}'(\zeta)-\mathbf{P}_{k}\boldsymbol{\Psi}(\zeta)=0,\quad \mathbf{U}[\boldsymbol{\Psi}(\zeta)]=\mathbf{I},\quad \bar{h}_{0}<\zeta<\bar{h}_{1} \tag{6.7}$$

To construct it [6, 7] we must have the solution $\mathbf{Y}(\zeta)$ of the matrix differential equation from (6.7). It can be shown that its solution will be the following integral [6, 7]

$$\mathbf{Y}(\zeta)=\frac{1}{2\pi i}\oint_{C}(\xi\mathbf{I}-\mathbf{P}_{k})^{-1}e^{\zeta\xi}d\xi \tag{6.8}$$

where the contour C envelopes all the zeros of the characteristic polynomial (6.2). Substituting expression (6.4) into (6.8) and taking the residues at multiple poles ν and $-\nu$, we obtain

$$\mathbf{Y}(\zeta)=\sum_{j=0}^{3}\Delta_{3-j}\left(\frac{d}{d\zeta}\right)^{j}\psi_{*}(\zeta),\quad \psi_{*}(\zeta)=\frac{\nu\zeta\,\mathrm{ch}\,\nu\zeta-\mathrm{sh}\,\nu\zeta}{2\nu^{3}} \tag{6.9}$$

It can be shown by direct substitution that the solution of boundary-value problem (6.7) is the matrix

$$\boldsymbol{\Psi}(\zeta)=\mathbf{Y}(\zeta)(\mathbf{U}[\mathbf{Y}(\zeta)])^{-1} \tag{6.10}$$

while Green's matrix function [6, 7] of boundary-value problem (5.8), (5.12) is the matrix

$$\mathbf{G}(\zeta,s)=\boldsymbol{\Phi}(\zeta-s)-\boldsymbol{\Psi}(\zeta)\mathbf{U}[\boldsymbol{\Phi}(\zeta-s)] \tag{6.11}$$

and thus [6, 7] its solution will be the vector

$$\mathbf{y}(\zeta)=\frac{\alpha_{\mu}}{\mu_{k}}\int_{\bar{h}_{0}}^{\bar{h}_{1}}\mathbf{G}(\zeta,s)\mathbf{f}(s)ds-(1-2\mu)\boldsymbol{\Psi}(\zeta)\boldsymbol{\gamma} \tag{6.12}$$

7. COMPLETION OF THE CONSTRUCTION OF THE EXACT SOLUTIONS OF THE PROBLEMS

Hence, the transform $\overline{W}_{nk}(\zeta)$ of the displacement

$$u_{z}(r,\varphi,z)=2GW(a_{1}\rho,\varphi,a_{1},\zeta)=2G\overline{W}(\rho,\varphi,\zeta)$$

and the transforms $\overline{Z}_{nk}(\zeta)$ and $\overline{Z}^{*}_{nk}(\zeta)\equiv 0$ are completely defined, but for the complete solution we need to obtain the transforms $\overline{U}_{nk}(\zeta)$ and $\overline{V}_{nk}(\zeta)$ of the displacements

$$u_{r}(r,\varphi,z)=2GU(a_{1},\rho,\varphi,a_{1}\zeta)=2G\overline{U}(\rho,\varphi,\zeta),\quad u_{\varphi}(r,\varphi,z)=2G\overline{V}(\rho,\varphi,\zeta)$$

To find them we will use Eqs (2.5). If we take equality (5.4) and the replacement (3.11) into account, as well as the notation

$$rY(r,\varphi,z)=a_{1}\rho\overline{Y}(\rho,\varphi,\xi)=a_{1}\overline{Y}^{*}(\rho,\varphi,\xi)$$

$$Y(r,\varphi,z)=\left\|\begin{matrix}U(r,\varphi,z)\\V(r,\varphi,z)\end{matrix}\right\|,\quad \overline{Y}(\rho,\varphi,\xi)=\left\|\begin{matrix}\overline{U}(\rho,\varphi,\xi)\\\overline{V}(\rho,\varphi,\xi)\end{matrix}\right\|,\ \text{etc.} \tag{7.1}$$

we can write the differential equations

$$\nabla \bar{Y}^*(\rho,\varphi,\zeta)=\left\| \begin{matrix} a_1\rho^{-1}[\rho^2\bar{Z}(\rho,\varphi,\zeta)]' \\ a_1\bar{Z}^*(\rho,\varphi,\zeta) \end{matrix} \right\| \tag{7.2}$$

With these equations we can set up boundary conditions so that the conditions of rigid clamping (the displacements are specified) on the faces $\varphi = \varphi_i$ $(i = 0, 1)$ are satisfied, i.e.

$$\bar{Y}^*(\rho,\varphi_i,\zeta)=0, \quad i=0,1 \tag{7.3}$$

These conditions will be satisfied if we apply integral transformation (3.1) to Eqs (7.2), i.e.

$$\bar{Y}_n^*(\rho,\zeta)=\int_0^{\varphi_1} \bar{Y}_n^*(\rho,\varphi,\zeta)\sin\mu_n\varphi d\varphi, \quad \mu_n=\frac{n\pi}{\varphi_1}, \quad n=1,2,\ldots,(\varphi_0=0) \tag{7.4}$$

As a result, Eqs (7.2) take the form

$$-\nabla_n\bar{Y}_n^*(\rho,\zeta)=\frac{a_1}{\rho}\left\| \begin{matrix} (\rho^2\bar{Z}_n)' \\ -\rho\mu_n\bar{Z}_n^c \end{matrix} \right\|, \quad \bar{Z}_n^c=\int_0^{\varphi_1}\bar{Z}(\rho,\varphi,\zeta)\cos\mu_n\varphi d\varphi \tag{7.5}$$

where in formula (3.10), which defines the operator ∇_n, we must take μ_n from (7.4) and replace r and z by ρ and ζ.

We must add to these differential equations the boundary conditions which ensure rigid clamping of the faces $r = a_i$ $(i = 0, 1)$, which, taking notation (7.1) and transformations (7.4) into account, can be written in the form

$$\bar{U}_n^*(\rho_i,\zeta)=\bar{V}_n^*(\rho_i,\zeta)=0, \quad i=0,1(\rho_0=a,\rho_1=1)$$

The latter will be satisfied if we use integral transformation (4.3) to solve Eqs (7.5) with $j = 0$, i.e.

$$\bar{Y}_{nk}^*(\zeta)=\int_a^1 \bar{Y}_n^*(\zeta)\varphi_{n,0}(\rho,\nu)\rho d\rho, \quad \nu=\nu_k^0, \quad k=1,2,\ldots \tag{7.6}$$

As a result we obtain

$$\bar{U}_{nk}^*(\zeta)=\frac{a_1}{\nu^2}\int_a^1 \bar{Z}_n\varphi_{n,0}'(\rho,\nu)\rho^2 d\rho, \quad \nu=\nu_k^0, \quad k=1,2,\ldots$$

$$\bar{V}_{nk}^*(\zeta)=\frac{a_1\mu_n}{\nu^2}\int_a^1 \bar{Z}_n^c(\rho,\zeta)\varphi_{n,0}(\rho,\nu)\rho d\rho, \quad \mu_n=\frac{n\pi}{\varphi_1} \tag{7.7}$$

Hence, the transforms of the displacements $\bar{U}_{nk}^*(\zeta)$, $\bar{V}_{nk}^*(\zeta)$ and $\bar{W}_{nk}(\zeta)$ (the latter is defined by (6.12) and (5.7)) are obtained, and from them, using the inversion formulae (4.6) and (3.2), and also formulae (2.1), (6.12) and (7.1), we obtain the displacements themselves, i.e. the exact solution of the problem in the case when the faces $r = a_i$, φ_i $(i = 0, 1)$ are rigidly clamped.

If, in conditions (1.1), $\varphi_0 = -\pi$ and $\varphi_1 = \pi$ and there are no faces $\varphi = \varphi_i$ $(i = 0, 1)$, then instead of transformations (7.4) we must apply to Eqs (7.2) the transformations

$$\bar{Y}_n^*(\rho,\zeta)=\frac{1}{2\pi}\int_{-\pi}^{\pi} \bar{Y}_n^*(\rho,\varphi,\zeta)e^{-in\varphi}d\varphi, \quad n=0,\pm1,\pm2,\ldots \tag{7.8}$$

As a result we arrive at Eqs (7.5), in which we must take $\bar{Z}_n^c = \bar{Z}_n$, $\mu_n = -in$, while the quantities $\bar{Z}_n$ must be taken in accordance with integral transformation (3.7), taking replacements (3.11) into account. Solving these equations using transformations (7.6) we obtain transforms (7.7), in which we must introduce these changes. The required displacements are found in the same way as in the preceding case, except that we must use formulae (3.8) instead of the inversion formulae (3.2).

We will consider the case of sliding clamping, i.e. when conditions (1.2) hold. The stresses there, according to notation (2.1), can be written in the form [2]

$$2r\tau_{\varphi r} = U^{\cdot} + r^2(r^{-1}\overline{V})', \quad 2r\tau_{\varphi z} = r\overline{V}^{\cdot} + \overline{W}^{\cdot}, \quad 2\tau_{rz} = W' + U^{\cdot}$$

or, in the notation of (3.11) and (7.1), in the form

$$2a_1\rho\tau_{\varphi r} = \overline{U}^{\cdot} + \rho\overline{V}' - \overline{V}, \; 2a_1\rho\tau_{\varphi z} = \rho\overline{V}^{\cdot} + \overline{W}^{\cdot}, \quad 2a_1\tau_{rz} = \overline{W}' + \overline{U}^{\cdot} \tag{7.9}$$

Hence, if we take into account the fact that the conditions $\overline{W}(\rho, \varphi_i, \zeta) = 0$ $(i = 0, 1)$ are already satisfied in view of conditions (3.6), the last three conditions of (1.2) will be satisfied as $\overline{U}(\rho, \varphi_i, \zeta) = \overline{V}(\rho, \varphi_i, \zeta) = 0$ $(i = 0, 1)$, or, in the notation of (7.1),

$$\overline{U}^{*}(\rho, \varphi_i, \zeta) = \overline{V}^{*}(\rho, \varphi_i, \zeta) = 0, \; i = 0, 1 \tag{7.10}$$

If we take into account the fact that, when finding $\overline{W}$ condition (4.2) is satisfied, then, to satisfy the first three conditions from (1.2) it is sufficient, by relations (7.9), to realise the conditions

$$\overline{U}(\rho_i, \varphi, \zeta) = 0, \quad [\rho\overline{V}' - \overline{V}]_{\rho=\rho_i} = 0, \quad i = 0, 1, \quad \rho_0 = a, \quad \rho_1 = 1$$

or in notation (7.1) the conditions

$$\overline{U}^{*}(\rho_i, \varphi, \zeta) = 0, \quad [\rho\overline{V}^{*\prime} - 2\rho^{-1}\overline{V}^{*}]_{\rho=\rho_i} = 0, \quad i = 0, 1, \quad \rho_0 = a, \quad \rho_1 = 1 \tag{7.11}$$

In order to satisfy conditions (7.10), we change in Eqs (7.2) to the respective transforms (everywhere henceforth $n, k = 1, 2, \ldots$)

$$\overline{U}_n^{*}(\rho, \zeta) = \int_0^{\varphi_1} \cos\mu_n\varphi\overline{U}^{*}(\rho, \varphi, \zeta)d\varphi, \quad \mu_n = (n-1)\pi\varphi_1^{-1}(\varphi_0 = 0) \tag{7.12}$$

$$\overline{V}_n^{*}(\rho, \zeta) = \int_0^{\varphi_1} \sin\mu_n\varphi\overline{V}^{*}(\rho, \varphi, \zeta)d\varphi, \quad \mu_n = n\pi\varphi_1^{-1}(\varphi_0 = 0) \tag{7.13}$$

Then Eqs (7.2) become

$$-\nabla\overline{U}_n^{*}(\rho, \zeta) = a_1\rho^{-1}[\rho^2\overline{Z}_n(\rho, \zeta)]', \quad \mu_n = (n-1)\pi\varphi_1^{-1} \tag{7.14}$$

$$-\nabla\overline{V}_n^{*}(\rho, \zeta) = a_1\mu_n\overline{Z}_n^0(\rho, \zeta), \quad \mu_n = n\pi\varphi_1^{-1} \tag{7.15}$$

Here $\overline{Z}_n$ is taken in accordance with (3.3), while for $\overline{Z}_n^0$ we take the same formula except that μ_n must be taken as in (7.13).

In order to satisfy the first condition of (7.11), in Eq. (7.14) we change to transforms given by formula (7.6)

$$\overline{U}_{nk}^{*}(\zeta) = \frac{a_1}{\nu^2}\int_a^1 \rho^2\overline{Z}_n(\rho, \zeta)\varphi_{n,0}'(\rho, \nu)d\rho, \quad \nu = \nu_k^0 \tag{7.16}$$

To satisfy the second condition of (7.11) we must apply to Eq. (7.15) the integral transformations [3]

$$\overline{V}_{nk}^{*}(\zeta) = \int_a^1 \rho\overline{V}_{nk}^{*}(\rho, \zeta)\varphi_{n*}(\rho, \nu)d\rho, \quad \nu = \nu_k^{*} \tag{7.17}$$

in which the kernel is given by the formula

$$a\varphi_{n*} = J_\mu(\nu\rho)[\nu aN_\mu'(a\nu) - N_\mu(a\nu)] - N_\mu(\nu\rho)[\nu aJ_\mu'(a\nu) - J_\mu(a\nu)], \quad \mu = \mu_n = n\pi\varphi_1^{-1}$$

where we take as ν the roots of the transcendental equation

$$[\nu J'_\mu(\nu) - J_\mu(\nu)][\nu a N'_\mu(a\nu) - N_\mu(a\nu)] - [\nu N'_\mu(\nu) - N_\mu(\nu)][\nu a J'_\mu(a\nu) - J_\mu(a\nu)] = 0,$$

$$\nu = \nu_k^* \tag{7.18}$$

As a result we obtain from Eq. (7.15)

$$\bar{V}_{nk}^*(\zeta) = \frac{a_1 \mu_n}{\nu^2} \int_0^{\varphi_l} \int_a^1 \rho \bar{Z}(\rho, \varphi, \zeta) \varphi_{n*}(\rho, \nu) \cos \mu_n \varphi d\varphi d\rho, \quad \nu = \nu_k^*, \quad \mu_n = n\pi\varphi_1^{-1} \tag{7.19}$$

The inversion formula for this transform has the form [3]

$$\bar{V}_n^*(\rho, \zeta) = \sum_{k=1}^{\infty} \frac{\varphi_{n*}(\rho, \nu)}{\| \varphi_{n*}(\rho, \nu) \|^2} \bar{\upsilon}_{nk}^*(\zeta) \tag{7.20}$$

$$2\nu^2 \| \varphi_{n*}(\rho, \nu) \|^2 = \nu^2 [\varphi'_{n*}(\nu, \nu)]^2 + (\nu^2 - \mu_n^2)\varphi_{n*}^2(\nu, \nu) - \nu^2 a^2 [\varphi'_{n*}(\nu, \nu)]^2 - (\nu^2 a^2 - \mu_n)^2 \times$$

$$\times \varphi_{n*}^2(\nu a, \nu), \quad \nu = \nu_k^*, \quad \mu = \mu_n = n\pi\varphi_1^{-1}$$

Hence, the transforms of the displacement $\bar{U}_{nk}^*(\zeta)$, $\bar{V}_{nk}^*(\zeta)$ and $\bar{W}_{nk}^*(\zeta)$ have been obtained and are defined by formulae (7.16), (7.19) and (6.12) and (5.7). From inversion formulae (7.20), (4.6) and (3.2), (3.4) we obtain their originals, and using formulae (2.1), (6.12) and (7.1) the displacements themselves, and we thereby obtain the exact solution of the problem for the case when the conditions of sliding clamping (1.2) are specified on the faces $r = a_i$, $\varphi = \varphi_i$ $(i = 0, 1)$.

If in conditions (1.1) $\varphi_0 = -\pi$ and $\varphi_1 = \pi$ and there are no faces $\varphi = \varphi_i$ $(i = 0, 1)$, we must apply integral transformation (7.8) to Eqs (7.2) instead of integral transformations (7.12) and (7.13). As a result they reduce to Eqs (7.14) and (7.15), in which we must take $\mu_n = -in$ while $\bar{Z}_n$ and $\bar{Z}_n^0$ are taken from formula (3.7). We apply integral transformations (7.6) and (7.17) to Eqs (7.14) and (7.15), modified in this way, where we must also take $\mu_n = n$ and also $\nu = \nu_n^0$, which are the roots of the first equations of (4.5) and $\nu = \nu_n^*$, which are the roots of Eq. (7.18). We must take $\mu = n$ in the latter equation. As a result we obtain the transforms $\bar{U}_{nk}^*(\zeta)$ and $\bar{V}_{nk}^*(\zeta)$, and, as in the previous case, we obtain the displacements, thereby completing the construction of the exact solution of the problem,

For the proposed method it is essential that the faces $r = a_i$, $\varphi = \varphi_i$ $(i = 0, 1)$ should either be rigidly clamped, or the conditions of sliding clamping should be satisfied on them, where, when describing the method, we assumed, in order to reduce the amount of calculation, that boundary conditions of the same type are satisfied on all the faces. The proposed constructions can easily be extended to the case when the boundary conditions are not of the same type. It is merely important that each time the form of the boundary conditions on the faces should not go beyond the limit stipulated above.

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Translated by R.C.G.